UNIT – I - Matrix & Vector Space

Reference text books:
1. Linear Algebra and Its Applications: by David C. Lay
https://math.berkeley.edu/~yonah/files/Linear%20Algebra.pdf
2. Linear Algebra: SCHAUM'S outlines
https://anujitspenjoymath.files.wordpress.com/2019/02/schaums-outline-series-lipschutz-seymour_-lipson-marc-schaums-outlines.-linear-algebra-2018-mcgraw-hill-education.pdf
http://www.astronomia.edu.uy/progs/algebra/Linear_Algebra,_4th_Edition__(2009)Lipschutz-Lipson.pdf

LINEAR ALGEBRA: Linear Algebra is the branch of mathematics concerning linear equations such as linear functions and their representations through matrices and vector spaces.

Applications of Linear Algebra:

Image processing (Image Representation as Tensors), Machine learning (Neural Network) Cryptography, Data structures, Gamming Technology and many more……

Linear algebra is made up of two basic elements: **The Matrix and the Vector.**

What is a Vector?

Vectors can be thought of as an array of numbers where the order of the numbers also matters.

Vectors in $\mathbb{R}^n$: The set of all n-tuples of real numbers, denoted by $\mathbb{R}^n$ is called *n-space.* A particular *n-tuple* in $\mathbb{R}^n$, say $u = (a_1, a_2, a_3, \ldots\ldots, a_n)$ is called a point or vector.

The following are vectors:

$$(2, -5), \quad (7, 9), \quad (0, 0, 0), \quad (3, 4, 5)$$

The first two vectors belong to $\mathbf{R}^2$, whereas the last two belong to $\mathbf{R}^3$. The third is the zero vector in $\mathbf{R}^3$.

A matrix with only one column(row) is called a **column(row) vector**, or simply a **vector.**

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1.5 \\ \frac{2}{3} \\ -15 \end{bmatrix}$$

Vector Addition and Scalar Multiplication

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

The *scalar product* or, simply, *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. The sum of vectors with different numbers of components is not defined.

EXAMPLE
Let $u = (2, 4, -5)$ and $v = (1, -6, 9)$. Then
$u + v = (2 + 1,\ 4 + (-6),\ -5 + 9) = (3, -2, 4)$
$7u = (7(2), 7(4), 7(-5)) = (14, 28, -35)$

Three kinds of mathematical structures

In order of increasing number of kinds of components:

Groups: one kind of element, one operation

Fields: one kind of element, two operations ("addition" and "multiplication")

Vector spaces: two kinds of elements (vectors and scalars); scalars form a field, and operations that apply to (vector, vector) pairs and to (vector, scalar) pairs

A **Group** G, sometimes denoted by $\{G, *\}$ is a set of elements with a binary operation, denoted by "$*$", that associates to each ordered pair (a, b) of elements in G an element $(a * b)$ in G, such that the following axioms are obeyed:

Closure: If a and b belong to G, then $a"*"\ b$ is also in G.

Associative: $a"*"(b"*"\ c) = (a"*"\ b)"*"\ c$ for all a, b, c in G.

Identity element: There is an element e in G such that $a * e = e * a = a$ for all a in G.

Inverse element: For each a in G there is an element a' in G such that $a * a' = a' * a = e$.

A group is said to be **Abelian** if it satisfies the following additional condition:

Commutative: $a * b = b * a$ for all a, b in G.

Examples:

- $\mathbb{Z}$, the set of integers, is an abelian group operation under addition.
- $\mathbb{R}$, the set of real numbers, is an abelian group operation under addition.
- $\mathbb{R} - \{0\}$, the set of non-zero real numbers, is an abelian group operation under multiplication.

A non- empty set F is called a **Field**, if :

- F is an abelian group under addition
- $F - \{0\}$ is an abelian group under multiplication.
- Right distributive law holds in F, i.e $a, b, c \in F$ then $(a+b)c = ac + bc$

Examples:

- $(\mathbb{R}, +, .)$ is a field
- $(\mathbb{Q}, +, .)$ is a field
- $(\mathbb{Z}, +, .)$ is **not a** field

VECTOR SPACE

A **vector space** is a **nonempty set** V of objects, called ***vectors***, and a Field F of scalars, on which are defined two operations, called ***addition*** and ***scalar multiplication***, subject to the **ten rules** listed below. These must hold for all vectors **u**, **v**, and **w** in V and for all scalars ***c*** and ***d***.

1. The sum of **u** and **v**, denoted by **u** + **v**, is in V. (Closed under addition)

2. **u + v = v + u**.

3. **(u + v) + w = u + (v + w)**

4. There is a **zero** vector **0** in V such that **u + 0 = u**.

5. For each **u** in V , there is a vector **-u** in V such that **u + (-u) = 0**.

6. The scalar multiple of **u** by ***c***, denoted by ***c*u**, is in V .

7. ***c*(u + v) = *c*u + *c*v**

8. **(*c* + *d*)u = *c*u + *d*u**.

9. ***c*(*d*u) = (*cd*)u.**

10. 1**u** = **u**.

Examples: The following are examples of vector spaces:

1. The set of all real number $\mathbb{R}$ with the addition and scalar multiplication of real numbers.
2. The set of all vectors of dimension *n* written as $\mathbb{R}^n$ associated with the addition and scalar multiplication as defined for 2-d($\mathbb{R}^2$) and 3-d($\mathbb{R}^3$) vectors for example.
3. The set of all polynomials $P_n(x) = \{a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n \;/\; a_i \in \mathbb{R}\}$ with real coefficients associated with the addition and scalar multiplication of polynomials.
   [eg. $P_3(x) = \{a_0 + a_1x + a_2x^2 + a_3x^n \;/\; a_i \in \mathbb{R}\}$.
   In this context, the ***0*** vector is $0 + 0x + 0x^2 + 0x^3 = 0$]